Generalized-Lorentzian path integrals

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A type of path integral is introduced that is based on q-Lorentzian functions. It extends the common family of Gaussian path integrals to a more general class with the Gaussian path integrals being the limiting case for $q\to 1$. Intuitively one may expect that this type of path integral applies to problems where inherent correlations become important. Application to turbulence or nonlinear Schrödinger problems may be possible. As a first step we provide the modified representation for transition probabilities and the generation functional. [S1063-651X(98)10905-4]

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Path integrals have been introduced into physics by Feynman [1] (for reference see also [2]) in an approach to reformulate quantum mechanics in terms of the many virtual orbits a particle may perform when passing from an initial position in spacetime to its final destination. These integrals have been based on Gaussian integrals because of their relatively easy mathematical treatment. In addition, however, the underlying Markovian assumption of equal probability of the particle orbits in quantum mechanics justified the Gaussian assumption and immediately lead to Gaussian statistics.

More recently it has been demonstrated that statistical mechanics may under some special, though so far not well understood, circumstances deviate from pure Markovian nature. The investigation of Lévy flight dynamics [3,4] suggests that statistical mechanics sometimes does not follow Gaussian distributions. Moreover, it has been suggested that thermodynamics can be reformulated to account for some observed nonextensivity [5]. This case has recently been given a sound statistical basis [6,7]. The important finding is that the Boltzmann-Gaussian distribution of states is replaced by a distribution of the type of a q-generalized Lorentzian

$$f_q(x)=A_q[1\pm(1-q)\beta x^2]^{\mp 1/(1-q)}, \tag{1}$$

where q is a parameter that in the classical case contains the underlying microscopic dynamics, β a constant corresponding to the temperature of the system, A_q an appropriate normalization constant, and the upper and lower signs refer to the two cases $q\leqslant 1$ and $q\geqslant 1$, respectively. In the limit of $q\to 1$ the distribution (1) smoothly approaches the Gaussian distribution of independent states. Hence, the above distribution function is a generalization of the usual Maxwell-Gaussian distribution, and q contains the information about the correlations between states. These correlations have been discussed in the context of some applications (see, e.g., [6]). An explicit expression for q applying to one particular case has been derived by Hasegawa *et al.* [8].

The above distribution gives the possibility to discuss thermodynamically nonextensive states and also to define a kind of quantum statistics whose properties differ from the known statistics [9]. Here we demonstrate that on its basis one can also define a new class of path integrals that may prove useful in treating statistical states with correlations. Such integrals can be constructed in analogy to the Gaussian path integrals by considering the non-Gaussian integral

$$\int_{-\infty}^{\infty} dx[1+(1-q)\beta x^2]^{-1/(1-q)}=\frac{2B[1/(1-q)-\frac{1}{2},\frac{1}{2}]}{[(1-q)\beta]^{1/2}}, \tag{2}$$

where $B(x,y)$ is the beta function, which is expressible through gamma functions. This integral converges for $|q|<1$. For $q\to +1$ it approaches its Gaussian value $(\pi\beta)^{-1/2}$. Though the above integration is easy to perform, there is a striking difference from the Gaussian limit in that convergence of the integral is strictly valid only for $|q|<1$ while for the related moment integrals where the integrand is multiplied by $x^{\alpha-1}$ convergence is restricted to the domain $1-q<2/\alpha$. Extension to the domain $q>1$ can thus be obtained only on the way of defining the complementary integral

$$\int_{-\infty}^{\infty} dx[1+(q-1)\beta x^2]^{-1/(q-1)}=\frac{2B[1/(q-1)-\frac{1}{2},\frac{1}{2}]}{[(q-1)\beta]^{1/2}}, \tag{3}$$

which exists for $1<q<3$ and in the limit $q\to +1$ also reproduces the Gaussian case. Negative values of q are excluded in this case.

We now generalize the integral in Eq. (2) to N dimensions. It then assumes the following compact form:

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_N}{\left[1+(1-q)\sum_{i=1}^{N}\beta_i x_i^2\right]^{1/(1-q)}} = \frac{2^N\prod_{i=1}^{N}B[1/(1-q)-\frac{1}{4}i(i+1),\frac{1}{2}]}{\prod_{i=1}^{N}[(1-q)\beta_i]^{1/2}}. \tag{4}$$

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This integral can be written as

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_N}{[1+(1-q)\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}]^{1/(1-q)}}$$
$$=\frac{2^N}{[(1-q)^N\det\mathsf{A}]^{1/2}}\prod_{i=1}^{N}B\left[\frac{1}{1-q}-\frac{1}{4}i(i+1),\frac{1}{2}\right].\quad(5)$$

The column vector $\mathbf{x}$ has elements x_i, the superscript T means taking the transposed, and β_i are the elements of the matrix A.

The last equation is similar to the result for Gaussian integrals. This is obvious because the Gaussian case is contained in our more general representation as the limit $q\to+1$. In order to make this similarity even more lucid we divide by the normalization factor on the right-hand side to obtain

$$\exp(-\mathrm{Tr}\ \ln\ \mathsf{A})=\frac{(1-q)^{N/2}}{2^N\prod_{i=1}^{N}B[1/(1-q)-\frac{1}{4}i(i+1),\frac{1}{2}]}\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_N}{[1+(1-q)\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}]^{1/(1-q)}}.\quad(6)$$

The identity of the exponential of the trace of the logarithm and the determinant is a well-known fact and can be proven by diagonalization. The left-hand side of this expression is finite for any dimension N of the system described by A, and hence the limit $N\to\infty$ of the right-hand side exists for any number N of variables. It is then possible to generalize it to continuous systems by introducing a field function $\phi(x)$ with x a continuous variable. This leads to the definition of a new type of path integral

$$\exp(-\mathrm{Tr}\ \ln\ \mathsf{A})=\int\mathcal{D}\phi(x)\left[1+(1-q)\int_{-\infty}^{\infty}dx'\times\int_{-\infty}^{\infty}dx\ \phi(x')A(x',x)\phi(x)\right]^{-1/(1-q)},\quad(7)$$

where

$$\mathcal{D}\phi(x)=\lim_{N\to\infty}\prod_{i=1}^{N}dx_i\frac{(1-q)^{N/2}}{2^N\prod_{l=1}^{N}B[1/(1-q)-\frac{1}{4}l(l+1),\frac{1}{2}]}\quad(8)$$

is the path differential that is defined as a limiting process.

With the formal definition of the path integral (7) we have extended the Gaussian path integrals to generalized Lorentzian path integrals. It can be easily shown that in the limit of $q\to+1$ they smoothly make the transition to Gaussian path integrals. It is hence clear that the Gaussian integrals are limiting cases of the above Lorentzian integrals. Since Gaussian processes are purely stochastic processes it can be argued that the new Lorentzian path integrals are appropriate for application in cases when deviations from stochasticity dominate the evolution of the system under consideration. The parameter q then provides a compactification of the nonstochastic correlation processes into one single number.

For applications of the path integral formalism it is very much desired to have an integral that contains a free variable. Such an integral can be constructed by adding a linear term to the square of the variable in the Lorentzian denominator in Eq. (6). We do this in the form

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_N}{\{1+(1-q)[\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}-2\vec{\rho}^T\cdot\mathbf{x}]\}^{1/(1-q)}}.\quad(9)$$

Here $\vec{\rho}$ is the new free N-dimensional column vector. This expression can be quadratically completed using

$$\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}-2\vec{\rho}^T\cdot\mathbf{x}=(\mathbf{x}-\mathsf{A}^{-1}\cdot\vec{\rho})^T\cdot\mathsf{A}\cdot(\mathbf{x}-\mathsf{A}^{-1}\cdot\vec{\rho})-\vec{\rho}^T\cdot\mathsf{A}^{-1}\cdot\vec{\rho}.\quad(10)$$

With the usual change of variables $\mathbf{x}'\equiv\mathbf{x}-\mathsf{A}^{-1}\cdot\vec{\rho}$ the integral including the free-parameter vector $\vec{\rho}$ can be solved yielding

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_N}{\{1+(1-q)[\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}-2\vec{\rho}^T\cdot\mathbf{x}]\}^{1/(1-q)}}=\frac{Q_N(q,\vec{\rho},\mathsf{A})\exp(-\mathrm{Tr}\ \ln\ \mathsf{A})}{[1-(1-q)\vec{\rho}^T\cdot\mathsf{A}^{-1}\cdot\vec{\rho}]^{1/(1-q)}},\quad(11)$$

where the factor $Q_N(q,\vec{\rho},\mathsf{A})$ is found to be

$$Q_N=\left[4\pi\left(\frac{1}{1-q}-\vec{\rho}^T\cdot\mathrm{A}^{-1}\cdot\vec{\rho}\right)\right]^{N/2}\times\prod_{i=1}^{N}\frac{\Gamma[1/(1-q)-\frac{1}{4}i(i+1)]}{\Gamma[1/(1-q)-\frac{1}{4}i(i+1)+\frac{1}{2}]}.$$

Q_N can be absorbed into the left-hand side integration when making the transition $N\to\infty$. One observes that the ratio of gamma functions under the product sign readily becomes one for sufficiently large numbers i such that the product contains only low numbers of the iterations.

Hence, dividing by Q_N and making the transition to very large N we define the extended path integral as

$$\int\frac{\mathcal{D}\phi(x)}{\{1+(1-q)\mathcal{A}[A,\phi]-2\mathcal{R}[\rho,\phi]\}^{1/(1-q)}}=\frac{\exp(-\mathrm{Tr}\ln\mathsf{A})}{\{1-(1-q)\mathcal{S}[A^{-1},\rho]\}^{1/(1-q)}},\quad(12)$$

with

$$\mathcal{A}[\mathsf{A},\phi]=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx\,\phi(x')A(x',x)\phi(x),$$

$$\mathcal{R}[\rho,\phi]=\int_{-\infty}^{\infty}dx\rho(x)\phi(x),$$

$$\mathcal{S}[\mathsf{A}^{-1},\rho]=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx\,\rho(x')A^{-1}(x',x)\rho(x).$$

Here $\rho(x)$ is a known function, and $A^{-1}(x',x)$ can be evaluated once the functional form of the matrix A is known. One simple choice of $A(x',x)$ is

$$A(x',x)=(\partial_{x'}\partial_x+a)\,\delta(x'-x), \tag{13}$$

in which case one obtains the usual representations

$$\mathrm{Tr}\;\ln\,\mathsf{A}=(2\pi)^{-1}\int dx\int dp\;\ln(p^2+a), \tag{14}$$

$$A^{-1}(x',x)=(2\pi)^{-1}\int dp\;\exp[ip(x'-x)]/(p^2+a).$$

The inclusion of the function $\rho(x)$ will allow for the definition of more general path integrals. One should note one distinction from the Gaussian case. The more complicated derivation of the integrals leads to a path element that is defined in a slightly different way because it now includes the arbitrary function ρ. This merely implies a different scaling of the length of the path element and does not affect the further development of the formalism.

Differentiating the integral in Eq. (12) with respect to the components of $\vec{\rho}$ at $\rho=0,\mathcal{A}=0$ one finds another class of integrals that can be transformed into path integrals. These integrals have the following form:

$$C_p(q)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{x_{m_1}\cdots x_{m_p}dx_1\cdots dx_N}{[1+(1-q)\mathbf{x}^T\cdot\mathsf{A}\cdot\mathbf{x}]^{p+1/(1-q)}}$$
$$=\exp(-\mathrm{Tr}\;\ln\,\mathsf{A})(A^{-1}_{m_1m_2}\cdots A^{-1}_{m_{p-1}m_p}+\mathrm{perm}). \tag{15}$$

Here perm means all possible permutations. Defining

$$M(p)=\sum_{i=1}^{p}m_i\,, \tag{16}$$

the constant factor $C_p(q)$ is given as

$$C_p(q)=\frac{2^{M(p)}}{1+(1-q)^{M(p)}}\,\frac{\Gamma(1/(1-q)+p)}{\Gamma(1/(1-q))}\left(\frac{1-q}{4\pi}\right)^{N/2}. \tag{17}$$

These integrals are nonzero for even p and vanish for odd p. They can immediately be rewritten as path integrals. Observing that $\Sigma_i m_i\rightarrow\Sigma_i i=p(p+1)/2$ in this case,

$$\int\frac{\mathcal{D}\phi\,\phi(x_1)\cdots\phi(x_p)}{\{1+(1-q)\mathcal{A}[\mathsf{A},\phi]\}^{1/(1-q)+p}}$$
$$=\frac{[1+(1-q)^{(p/2)\,(p+1)}]}{\exp(\mathrm{Tr}\;\ln\,\mathsf{A})}$$
$$\times[A^{-1}(x_1,x_2)\;\ldots A^{-1}(x_{p-1},x_p)+\mathrm{perm}]. \tag{18}$$

Except for the denominator, the factor $C_p(q)$ has been included into the limiting process, which leads to the definition of the path element $\mathcal{D}\phi$. Again, because of the symmetry this integral exists only for even values of p and vanishes for odd p. The derivatives in the path integral formalism are understood as functional derivatives:

$$\frac{\delta}{\delta\rho(x_i)}\int dx\,\rho(x)\phi(x)=\phi(x_i),\quad i=1,\ldots,p. \tag{19}$$

Generalization of the formalism to complex column vectors $\mathbf{z}=(z_1,\ldots,z_N),\mathbf{z}^\dagger=\mathbf{z}^{*T}$ and Hermitian matrices A is straightforward. In this case it holds that $\int dzdz^*=2\int d(\mathrm{Re}\,z)d(\mathrm{Im}\,z)$, and one finds for the corresponding path integral

$$\int\int\frac{\mathcal{D}\phi\mathcal{D}\phi^*}{\{1+(1-q)\tilde{\mathcal{A}}[\mathsf{A},\phi,\phi^*]\}^{1/(1-q)}}=\exp(-\mathrm{Tr}\;\ln\,\mathsf{A}), \tag{20}$$

where we have defined

$$\tilde{\mathcal{A}}[\mathsf{A},\phi,\phi^*]=\int dx'\int dx\,\phi^*(x')A(x',x)\phi(x). \tag{21}$$

From this expression a complex generalization of the functional derivative path integral (18) automatically follows.

This further generalization completes our discussion, which was restricted to the case $q<1$. Extension to the complementary case $q>1$ is trivially done by replacing $1-q$ with $q-1$ in all expressions. But since the domains of definition are not symmetric for both cases, the resulting integrals will have different properties and will apply to different physical conditions.

In summary, we have demonstrated that it is possible to define a class of path integrals that are non-Gaussian. These path integrals can be understood as describing non-Gaussian processes containing nonvanishing correlations among states of a system as is suggested by statistical mechanical investigations of the generalized Lorentzian distribution function (1) [5,6,7]. It will be interesting to explore what effects can be described in the language of such path integrals and which quantum theoretical problems can be solved by its application. Formally, this opens up the possibility to reformulate quantum mechanics in terms of such integrals. This requires further investigation of the physical relevance of correlations in the quantum domain, a problem of interest in the attempts to detect quantum chaos.

In order to provide a first physical application we refer to the definition of the path integral representation of transition amplitudes between two states $Q'=Q(t'),Q''=Q(t'')$ for a system described by the (classical) Hamiltonian $H(P,Q)$, with Q,P the canonically conjugate positions and momenta, respectively. The well-known expression for the transition amplitudes is

$$\langle Q'',t''|Q',t'\rangle\propto\int\mathcal{D}Q\mathcal{D}P\;\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}(P\dot{Q}-H)dt\right]. \tag{22}$$

In the above case the transition is described by the linear Schrödinger equation. For Hamiltonians of the form $H(P,Q)=P^2/2m+V(Q)$, with m the particle mass, $V(Q)$ the generalized scalar potential, and making the transition to imaginary time $it \to \overline{t}$, the above integral becomes Gaussian. This suggests that replacing the exponential function with the corresponding Lorentzian will describe transitions in a *nonlinear* Schrödinger system where the different states are not independent (as in nonlinear laser interaction). We may formally write such an integral as

$$\langle Q'',t''|Q',t'\rangle \propto \int \mathcal{D}Q\mathcal{D}P\left[1+\frac{1-q}{\hbar}\times\int_{\overline{t'}}^{\overline{t''}} d\overline{t}(iP\dot{Q}-H+JQ)\right]^{-1/(1-q)}, \tag{23}$$

where an external driving force $J(t)Q$ has been introduced. Clearly, in this case the linear relation $\hat{S}|Q\rangle = Q|Q\rangle$, with $\hat{S}$ the Schrödinger operator and Q its eigenvalue is not applicable. Instead, the new definition of the expectation value is

$$\langle Q'',t''|Q',t'\rangle = \langle Q''|\left[1+\frac{i(1-q)}{\hbar}\times\hat{H}(t''-t')\right]^{-1/(1-q)}|Q'\rangle. \tag{24}$$

$\hat{H}$ is the (time independent) Hamilton operator. Formally it is also possible to write down the ground-to-ground state amplitude $W[J]$ in the presence of external force fields for the Lorentzian case:

$$W[J] \propto \int \mathcal{D}Q\mathcal{D}P\left[1+\frac{1-q}{\hbar}\times\int_{-\infty}^{\infty} d\overline{t}\left(H-iP\frac{dQ}{d\overline{t}}-JQ\right)\right]^{-1/(1-q)}. \tag{25}$$

From here the transition to nonlinear scalar field theory is straightforward. Defining the field function and momentum by φ, ϖ, respectively, the field generating functional reads

$$\mathcal{W}[J] = K\int \mathcal{D}\varphi\mathcal{D}\varpi\left[1+\frac{1-q}{\hbar}\times\int d^4\overline{x}\left(\mathcal{H}-i\varpi\frac{\partial\varphi}{\partial\overline{x}_0}-J\varphi\right)\right]^{-1/(1-q)}, \tag{26}$$

with Hamiltonian density $\mathcal{H}(\varphi,\varpi)$ and boundary conditions

$$\lim_{\overline{x}_0\to\infty}\varphi(\overline{x}) = \varphi'(\mathbf{x}), \quad \lim_{\overline{x}_0\to-\infty}\varphi(\overline{x}) = \varphi''(\mathbf{x}), \tag{27}$$

where we have used the four-dimensional Minkowski notation with x_0 the time coordinate. In addition the normalization constant K is chosen such that $\mathcal{W}[J]=1$ for $J=0$. These generating functionals serve as generators for the Green functions of the particular problem for given control parameter q. As usual, knowledge of the Hamiltonian density and the external driving forces is required.

Nonlinear Schrödinger equations are known from nonlinear optics, nonlinear plasma, and turbulence theories where they describe phenomena like soliton formation, plasma collapse, nonlinear laser interaction, anomalous absorption of radiation, chaotic transitions in pumped lasers, and transition to turbulence. In all these processes correlations are of fundamental importance.

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